

UNITARY REPRESENTATIONS OF THE GROUPS OF MEASURABLE AND CONTINUOUS FUNCTIONS WITH VALUES IN THE CIRCLE

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ABSTRACT. We give a classification of unitary representations of certain Polish, not necessarily locally compact, groups: the groups of all measurable functions with values in the circle and the groups of all continuous functions on compact, second countable, zero-dimensional spaces with values in the circle. In the proofs of our classification results, certain structure theorems and factorization theorems for linear operators are used.

1. INTRODUCTION

We study unitary representations of the groups of measurable and continuous functions with values in the circle. A description of unitary representations of such groups is of interest especially in view of recent considerable activity around topological and measurable dynamics of these groups; see the remarks below. The reader may consult [14] for background information on dynamics of Polish non-locally compact groups. (Recall that a topological group is *Polish* if its topology is metrizable by a complete separable metric.)

For a Borel probability measure μ on a standard Borel space and a topological group H , let $L^0(\mu, H)$ be the topological group of all μ -equivalence classes of μ -measurable functions with values in H . The multiplication on $L^0(\mu, H)$ is implemented pointwise and the topology is the convergence in measure topology. Groups of this form were perhaps first systematically considered in [3] to provide an embedding of each topological group into a connected group. Recently, the more particular Polish groups $L^0(\mu, \mathbb{T})$ generated substantial interest in the context of extreme amenability [1], measure preserving group actions [2], representation properties of Polish groups [15], and generic properties of monothetic subgroups of certain large groups [9], [17].

2000 *Mathematics Subject Classification.* 22A25, 46A16, 46E10, 46E30.

Key words and phrases. Polish group, group of continuous functions, group of measurable functions, unitary representation.

Research supported by NSF grant DMS-1001623.

We also consider groups $C(M, \mathbb{T})$ of all continuous functions from M to the circle group \mathbb{T} , where M is a compact, second countable space. We take $C(M, \mathbb{T})$ with the pointwise multiplication and with the topology of uniform convergence. This arrangement makes $C(M, \mathbb{T})$ into a Polish group. Recently, measure preserving actions of $C(M, \mathbb{T})$ were studied in [11]. Observe, that groups described above are usually not locally compact: $L^0(\mu, \mathbb{T})$ is not locally compact when μ is not purely atomic and neither is $C(M, \mathbb{T})$ when M is infinite.

The goal of the present paper is to give classifications of strongly continuous unitary representations of the Polish groups $L^0(\mu, \mathbb{T})$ and $C(M, \mathbb{T})$ when M is zero-dimensional; see Theorems 2.1 and 3.1. (Recall that a space is *zero-dimensional* if it has a topological basis consisting of sets that are both closed and open; a typical example is the Cantor set.) Roughly speaking, the classification results say that the groups $L^0(\mu, \mathbb{T})$ and $C(M, \mathbb{T})$ behave like infinite dimensional tori. Despite the fact that these groups do not have irreducible unitary representations of dimension greater than 1 (even of dimension greater than 0 in the case of $L^0(\mu, \mathbb{T})$ for μ without atoms), their unitary representations can be constructed as *countable* direct sums of simple building blocks with the blocks being *uniquely determined* by the representation; see Subsections 2.1 and 3.1 for a description of these blocks.

In the proof of the classification result for $L^0(\mu, \mathbb{T})$, Theorem 2.1, besides the spectral theorem, an important role is played by Kwapien's structure theorem for linear operators between linear L^0 spaces. One consequence of our result, Corollary 2.2, is the existence, for a given continuous unitary representation of $L^0(\mu, \mathbb{T})$, of a unique up to measure equivalence, smallest with respect to the relation of absolute continuity, finite Borel measure ν that is absolutely continuous with respect to μ and is such that the representation is the composition of the natural homomorphism $L^0(\mu, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T})$ and a continuous unitary representation of $L^0(\nu, \mathbb{T})$. The proof of the classification result for $C(M, \mathbb{T})$, Theorem 3.1, uses the classification of unitary representations of $L^0(\mu, \mathbb{T})$ and certain factorization theorems for linear operators. As a consequence of Theorem 3.1, we get that given a continuous unitary representation of $C(M, \mathbb{T})$ there exists a unique up to measure equivalence, smallest with respect to the relation of absolute continuity, finite Borel measure ν on M such that the representation is the composition of the natural homomorphism $C(M, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T})$ and a continuous unitary representation of $L^0(\nu, \mathbb{T})$. This result is included in Corollary 3.2.

Notation and conventions. By \mathbb{R} , \mathbb{C} , and \mathbb{T} we denote the real numbers, the complex numbers, and the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$, respectively. The following spaces will be involved in our considerations:

$C(M, G)$, $L^p(\mu, G)$, where M is a compact metrizable space, $p = 0, 2$, μ is a Borel probability measure on a standard Borel space, and $G = \mathbb{T}, \mathbb{R}, \mathbb{C}^k$, for $k \in \mathbb{N}$. When $G = \mathbb{T}$ these spaces will be regarded as groups, when $G = \mathbb{R}$ or $G = \mathbb{C}^k$ they will be regarded as linear spaces over \mathbb{R} or \mathbb{C} , respectively. Unless otherwise stated (and this option will be exercised), $C(M, G)$ is equipped with the uniform convergence topology, $L^0(\mu, G)$ with the convergence in measure topology, and $L^2(\mu, G)$ with the L^2 topology. Note however that on $L^0(\mu, \mathbb{T})$ the convergence in measure and the L^2 topology coincide. The unitary group of a complex Hilbert space H will be denoted by $\mathcal{U}(H)$ and it will always be considered with its strong operator topology.

I thank Ilijas Farah for our discussions of extreme amenability in 2004, Marius Junge for pointing out to me Pisier's book [16], and Vladimir Pestov for valuable comments.

2. UNITARY REPRESENTATIONS OF $L^0(\mu, \mathbb{T})$

Fix a Borel probability measure μ on a standard Borel space X . We are interested in describing all continuous unitary representations of $L^0(\mu, \mathbb{T})$.

Fix a linear order $<_X$ on X of which we will assume that the order topology it generates is compact, second countable and the Borel sets with respect to this topology coincide with the Borel sets on X . For example, we can fix a Borel isomorphism from X to a closed subset of $[0, 1]$ and use it to pull back the standard linear order on $[0, 1]$. The order is important for the uniqueness part of Theorem 2.1.

2.1. Description of representations. Assume that we are given a sequence $\kappa = (k_1, \dots, k_n)$ of elements of $\mathbb{Z} \setminus \{0\}$ with

$$(1) \quad k_1 \leq k_2 \leq \dots \leq k_n.$$

Assume we have a finite Borel measure λ on X^n whose marginal measures are absolutely continuous with respect to μ , that is, for $i \leq n$

$$(A1) \quad (\pi_i)_*(\lambda) \ll \mu,$$

where, for $i \leq n$, $\pi_i: X^n \rightarrow X$ is the projection on the i -th coordinate. With this set of data we associate the following representation of $L^0(\mu, \mathbb{T})$ on $L^2(\lambda, \mathbb{C})$:

$$L^0(\mu, \mathbb{T}) \ni f \rightarrow U_f \in \mathcal{U}(L^2(\lambda, \mathbb{C})),$$

where for $h \in L^2(\lambda, \mathbb{C})$

$$U_f(h) = \left(\prod_{i \leq n} (f \circ \pi_i)^{k_i} \right) h.$$

Thus, the bounded function $\prod_{i \leq n} (f \circ \pi_i)^{k_i}$ acts on $h \in L^2(\lambda, \mathbb{C})$ by multiplication. Condition (A1) ensures that the representation is well defined. We

denote this representation by $\sigma(\kappa, \lambda)$. We do allow λ to be the zero measure, in which case $\sigma(\kappa, \lambda)$ is the trivial representation.

We will consider the following two additional conditions on the finite measure λ as above: for $1 \leq i < j \leq n$

$$(A2) \quad \lambda(\{(x_1, \dots, x_n) \in X^n : x_i = x_j\}) = 0,$$

and for $1 \leq i < j \leq n$ with $k_i = k_j$,

$$(A3) \quad \lambda(\{(x_1, \dots, x_n) \in X^n : x_j <_X x_i\}) = 0,$$

Conditions (A2) and (A3) are needed for the uniqueness part of the theorem below.

2.2. Statement of the main result. Let S be the set of all sequences $\kappa = (k_1, \dots, k_n)$ of elements of $\mathbb{Z} \setminus \{0\}$ with property (1). We say that the natural number n in $\kappa = (k_1, \dots, k_n)$ is the length of κ and denote it by $|\kappa|$.

Theorem 2.1. *Let ϕ be a continuous unitary representation of $L^0(\mu, \mathbb{T})$ on a separable complex Hilbert space H . Consider H_0 , the orthogonal complement of*

$$\{v \in H : \forall f \in L^0(\mu, \mathbb{T}) \phi(f)(v) = v\}.$$

Existence: For $\kappa \in S$ and $i \in \mathbb{N}$, there exist finite Borel measures λ_κ^i on $X^{|\kappa|}$ with properties (A1), (A2), (A3), and with

$$(A4) \quad \lambda_\kappa^j \ll \lambda_\kappa^i, \text{ for } i < j,$$

such that the representation ϕ restricted to H_0 is the direct sum of the representations $\sigma(\kappa, \lambda_\kappa^i)$ with $\kappa \in S$ and $i \in \mathbb{N}$.

Uniqueness: If the restriction of ϕ to H_0 is presented as the direct sum of $\sigma(\kappa, (\lambda')_\kappa^i)$, for $\kappa \in S$ and $i \in \mathbb{N}$, with (A1), (A2), (A3), and (A4), then, for each i and κ , λ_κ^i and $(\lambda')_\kappa^i$ are absolutely continuous with respect to each other.

We point out the following corollary. Recall first that if μ and ν are finite Borel measures on a standard Borel space X with $\nu \ll \mu$, then there is the natural surjective homomorphism $L^0(\mu, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T})$; simply note that the ν -equivalence class of a Borel function from X to \mathbb{T} contains its μ -equivalence class.

Corollary 2.2. *Let μ be a Borel probability measure on a standard Borel space X . Let ϕ be a continuous unitary representation of $L^0(\mu, \mathbb{T})$. There exists a finite Borel measure ν on X such that*

- (i) $\nu \ll \mu$ and ϕ is the composition of the natural homomorphism from $L^0(\mu, \mathbb{T})$ to $L^0(\nu, \mathbb{T})$ and a continuous unitary representation of $L^0(\nu, \mathbb{T})$;

- (ii) if ν' is a finite Borel measure on X with $\nu' \ll \mu$ and such that ϕ is the composition of the natural homomorphism from $L^0(\mu, \mathbb{T})$ to $L^0(\nu', \mathbb{T})$ and a continuous unitary representation of $L^0(\nu', \mathbb{T})$, then $\nu \ll \nu'$.

Proof. We keep the notation from Theorem 2.1. Let λ_κ^j , $\kappa \in S$, $i \in \mathbb{N}$, give a presentation of ϕ restricted to H_0 . For each $\kappa \in S$ consider the $|\kappa|$ many projections $X^{|\kappa|} \rightarrow X$ and form push-forward measures on X using these projections and measures λ_κ^j as i varies over \mathbb{N} . As κ varies over S , this procedure gives countably many finite Borel measures on X . Form their weighted sum with positive coefficients to obtain a finite Borel measure ν on X . This measure clearly fulfils (i).

To see (ii), fix ν' as in the assumption. Consider the continuous unitary representation of $L^0(\nu', \mathbb{T})$ as in this assumption. The presentation of this representation on H_0 as in Theorem 2.1 is also a presentation of ϕ . By the uniqueness part of Theorem 2.1, the corresponding measures in these two presentations are absolutely continuous with respect to each other. It follows that the push-forward of each λ_κ^j by each projection as in the previous paragraph is absolutely continuous with respect to ν' and (ii) now follows by our definition of ν . \square

2.3. Background results. We state here three results that will be needed in the sequel. The first one is a factorization result, the second is the form of the spectral theorem needed in this paper, and the third is a structure theorem for certain linear operators.

The first result is implicit in the proof of [4, Theorem 5]; see page 208 of [4].

Proposition 2.3. *Let ν be a probability measure on a standard Borel space. Let G be a real topological vector space that is separable and completely metrizable. Then for each continuous homomorphism $\phi: G \rightarrow L^0(\nu, \mathbb{T})$ there exists a continuous linear operator $\theta: G \rightarrow L^0(\nu, \mathbb{R})$ such that*

$$\phi = \exp(2\pi i\theta).$$

The following result is the version of the spectral theorem that we need later on; see [13, Proposition 4.7.13].

Proposition 2.4. *Let \mathcal{T} be a family of commuting unitary operators on a separable complex Hilbert space H . Then there exist a Borel probability measure ν on a standard Borel space and a surjective isometric operator $U: H \rightarrow L^2(\nu, \mathbb{C})$ such that for each $T \in \mathcal{T}$, UTU^{-1} is the multiplication operator on $L^2(\nu, \mathbb{C})$ by an element of $L^0(\nu, \mathbb{T})$.*

The following theorem of Kwapien [7] (see also [5, Theorem 8.4]) describing the structure of continuous linear operators from L^0 to L^0 is important in our argument. Recall that a measurable function σ between two standard Borel spaces with Borel probability measures is called *non-singular* if preimages under σ of measure zero sets are of measure zero.

Proposition 2.5. *Let μ, ν be Borel probability measures on standard Borel spaces X and Y , respectively. Let $T: L^0(\mu, \mathbb{R}) \rightarrow L^0(\nu, \mathbb{R})$ be a continuous linear function. Then there exist non-singular maps $\sigma_n: Y \rightarrow X$ and $g_n \in L^0(\nu, \mathbb{R})$, with $g_n(x) = 0$ holding for ν -almost all $x \in Y$ and for all but finitely many n , such that for $f \in L^0(\mu, \mathbb{R})$ we have*

$$T(f) = \sum_{n \in \mathbb{N}} g_n \cdot (f \circ \sigma_n).$$

2.4. Proof of Theorem 2.1. In this proof, when we say “representation” we mean “strongly continuous unitary representation.” In the proof, we write $L^2(\nu)$ for $L^2(\nu, \mathbb{C})$.

Proof of existence of the presentation. Assume we have a representation

$$\phi: L^0(\mu, \mathbb{T}) \rightarrow \mathcal{U}(H)$$

on a separable complex Hilbert space H . We divide the proof into three steps.

Step 1. We show that for each closed non-trivial subspace H_1 included in H_0 (as defined in the theorem) and invariant under the representation, there exists a closed non-trivial subspace H' of H_1 invariant under the representation such that the representation restricted to H' is of the form $\sigma(\kappa, \lambda)$ for some $\kappa \in S$ and some λ fulfilling (A1) and (A2). (Condition (A3) will be dealt with in Step 2.)

We note that H_0 is invariant under the representation as is its orthogonal complement. Therefore, for simplicity of notation, we can assume that $H_0 = H$ and, in fact, for the same reason, we assume that $H_1 = H$. Since $L^0(\mu, \mathbb{T})$ is abelian, it follows from the spectral theorem, Proposition 2.4, that there exists a Borel probability measure ν on a standard Borel space Y such that the representation ϕ is of the form

$$L^0(\mu, \mathbb{T}) \ni f \rightarrow N_f \in \mathcal{U}(L^2(\nu))$$

where

$$N_f(h) = \psi(f) \cdot h$$

for some continuous homomorphism

$$\psi: L^0(\mu, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T}).$$

By precomposing ψ with the exponential homomorphism

$$L^0(\mu, \mathbb{R}) \rightarrow L^0(\mu, \mathbb{T}); f \rightarrow \exp(2\pi i f),$$

we obtain a continuous homomorphism $\psi': L^0(\mu, \mathbb{R}) \rightarrow L^0(\nu, \mathbb{T})$. By Proposition 2.3, there exists a continuous linear operator

$$\theta: L^0(\mu, \mathbb{R}) \rightarrow L^0(\nu, \mathbb{R})$$

such that

$$\psi' = \exp(2\pi i \theta).$$

Using Kwapien's theorem, Proposition 2.5, applied to the operator θ , we find ν -measurable functions $g_n: Y \rightarrow \mathbb{R}$ and non-singular functions $\sigma_n: Y \rightarrow X$, $n \in \mathbb{N}$, such that for ν -almost all $x \in Y$ only finitely many $g_n(x)$ are non-zero and

$$\theta(f) = \sum_n g_n \cdot (f \circ \sigma_n).$$

It follows that for each $f \in L^0(\mu, \mathbb{R})$ we have

$$\psi(\exp(2\pi i f)) = \exp(2\pi i \sum_n g_n \cdot (f \circ \sigma_n)).$$

The above equality implies that for $f \in L^0(\mu, \mathbb{R})$

$$(2) \quad f \text{ has integer values} \implies \sum_n g_n \cdot (f \circ \sigma_n) \text{ has integer values.}$$

Now we partition Y up to a ν -measure zero set into countably many ν -measurable sets A for which there is a finite set $D \subseteq \mathbb{N}$ depending on A such that for every $y \in A$

$$D = \{n: g_n(y) \neq 0\}$$

and for all $y, y' \in A$ and $m, n \in D$

$$(3) \quad \sigma_m(y) = \sigma_n(y) \iff \sigma_m(y') = \sigma_n(y').$$

This is done as follows. Each finite $D \subseteq \mathbb{N}$ determines a ν -measurable set

$$A_D = \{y \in Y: \{n: g_n(y) \neq 0\} = D\}.$$

Each $y \in A_D$ induces an equivalence relation E_y on D by the formula

$$mE_y n \iff \sigma_m(y) = \sigma_n(y), \text{ for } m, n \in D.$$

There are finitely many equivalence relations on the finite set D . Therefore, we can partition A_D into finitely many ν -measurable sets by putting $y, y' \in A_D$ in the same set precisely when $E_y = E_{y'}$. These are the sets A of our countable partition. It is clear that they are disjoint, they cover each A_D , and there is countably many of them. Note further that with each such set A we can associate an equivalence relation E on D given by $E = E_y$ for each, equivalently any, $y \in A$.

We fix for a moment A , D and E as above. Let $D' \subseteq D$ pick precisely one point from each E -equivalence class. Then for each $f \in L^0(\mu, \mathbb{R})$

$$(4) \quad \sum_{n \in D} g_n \cdot (f \circ \sigma_n) \upharpoonright A = \sum_d \left(\left(\sum_{n \in d} g_n \right) \cdot (f \circ \sigma_{n_d}) \upharpoonright A \right),$$

where d runs over the set of all E -equivalence classes and n_d is the unique element of $D' \cap d$. On the set A , for $n \in \mathbb{N}$, define

$$k_n = \begin{cases} \sum_{m \in d} g_m, & \text{if } n = n_d \text{ for an } E\text{-equivalence class } d; \\ 0, & \text{if } n \notin D'. \end{cases}$$

It follows from (4) that

$$(5) \quad \sum_n g_n \cdot (f \circ \sigma_n) \upharpoonright A = \sum_{n \in D} g_n \cdot (f \circ \sigma_n) \upharpoonright A = \sum_{n \in D'} k_n \cdot (f \circ \sigma_n) \upharpoonright A.$$

Note also that by definition of E and D' (so ultimately by (3)) for distinct $m, n \in D'$ and ν -almost every $y \in A$, we have

$$(6) \quad \sigma_m(y) \neq \sigma_n(y).$$

We claim that A can be covered by countably many ν -measurable sets A_l , $l \in \mathbb{N}$, for which there are μ -measurable sets $B_l^n \subseteq X$ with $n \in D'$ so that for each $l \in \mathbb{N}$ and for distinct $m, n \in D'$ we have

$$(7) \quad B_l^m \cap B_l^n = \emptyset \quad \text{and} \quad \sigma_n(A_l) \subseteq B_l^n.$$

To see this, cover

$$X^{D'} \setminus \bigcup_{m \neq n, m, n \in D'} \{(x_i)_{i \in D'} \in X^{D'} : x_m = x_n\}$$

with sets enumerated by $l \in \mathbb{N}$ and of the form

$$\prod_{m \in D'} B_l^m$$

with $B_l^n \subseteq X$ μ -measurable and with

$$B_l^m \cap B_l^n = \emptyset$$

for all l and for distinct $m, n \in D'$. Now let

$$A_l = \bigcap_{n \in D'} \sigma_n^{-1}(B_l^n).$$

Condition (6) ensures that A is covered by the sets A_l . These sets are easily seen to be as required by (7).

Fix $l \in \mathbb{N}$ and $n_0 \in D'$. Since the function $\chi_{B_l^{n_0}} \in L^0(\mu, \mathbb{R})$ has integer values, by (2), we see that

$$\sum_n g_n \cdot (\chi_{B_l^{n_0}} \circ \sigma_n)$$

has integer values as well. But by (5) and by (7), we have

$$\sum_n g_n \cdot (\chi_{B_l^{n_0}} \circ \sigma_n) \upharpoonright A_l = \sum_{n \in D'} k_n \cdot (\chi_{B_l^{n_0}} \circ \sigma_n) \upharpoonright A_l = k_{n_0} \upharpoonright A_l.$$

Thus, $k_{n_0} \upharpoonright A_l$ has integer values. Since this is true for each $l \in \mathbb{N}$, we see that $k_{n_0} \upharpoonright A$ has integer values. This statement holds for all $n_0 \in D'$. Since $k_n \upharpoonright A = 0$ for $n \notin D'$, we see that all k_n have integer values on A .

Since the sets A partition Y up to a set of ν -measure zero, the functions k_n are defined ν -almost everywhere, they are integer valued, and, from (5), they fulfill

$$\sum_n g_n \cdot (f \circ \sigma_n) = \sum_n k_n \cdot (f \circ \sigma_n).$$

It follows that

$$(8) \quad \psi(\exp(2\pi i f)) = \prod_n \exp(2\pi i k_n \cdot (f \circ \sigma_n)) = \prod_n (\exp(2\pi i f) \circ \sigma_n)^{k_n}.$$

Finally, partition A into countably many ν -measurable sets B such that each k_n with $n \in D'$ is constant on each of these sets. Then by (6) and (8) it follows that the partition of Y into such sets B gives a decomposition of $L^2(\nu)$ into orthogonal subspaces of the form $L^2(\nu \upharpoonright B)$ that are invariant under the representation and are such that the representation on each of them has the following form. There exist a sequence $k_1 \leq \dots \leq k_n$ of integers and Borel non-singular functions $\sigma_i: B \rightarrow X$, $i \leq n$, with

$$(9) \quad \sigma_i(y) \neq \sigma_j(y), \text{ for } i \neq j \text{ and } y \in B$$

such that under the representation the operator associated with $f \in L^0(\mu, \mathbb{T})$ is given by

$$L^2(\nu \upharpoonright B) \ni h \rightarrow \left(\prod_{i \leq n} (f \circ \sigma_i)^{k_i} \right) h.$$

Since the representation is assumed to be non-trivial (as H_1 is assumed non-trivial), there is at least one set $B \subseteq Y$ in the decomposition above such that the sequence k_1, \dots, k_n associated with it is not constantly equal to 0. Let $\kappa = (l_1, \dots, l_m)$ be obtained from k_1, \dots, k_n by deleting all 0-s, and let $l_1 \leq \dots \leq l_m$. Note that $\kappa \in S$. Further, let τ_1, \dots, τ_m list the σ_i -s with i -s not corresponding to $k_i = 0$. Note that

$$\tau = (\tau_1, \dots, \tau_m) : B \rightarrow X^{|\kappa|}.$$

Let λ be the measure on $X^{|\kappa|}$ obtained from μ by pushing it forward by τ . Note that non-singularity of each τ_i implies that condition (A1) holds. Condition (9) implies (A2). Consider the closed space of all elements of $L^2(\nu \upharpoonright B)$ that are constant on the preimages under τ of λ almost all points in $X^{|\kappa|}$. (One makes this statement precise, as usual, by disintegrating $\nu \upharpoonright B$

with respect to τ .) Note that this space is invariant under the representation, is non-trivial, and the representation on it is of the form $\sigma(\kappa, \lambda)$.

Thus, as required, we produced a non-trivial subspace H' that is invariant under the representation and the representation restricted to H' is unitarily equivalent to $\sigma(\kappa, \lambda)$.

Step 2. We show here that ϕ restricted to H_0 is a direct sum of countably many representation of the form $\sigma(\kappa, \lambda)$ for some $\kappa \in S$ and λ with (A1), (A2), and also (A3). First, note that Zorn's lemma allows us to pick a maximal family \mathcal{F} of mutually orthogonal non-trivial subspaces H' of H_0 such that the representation restricted to each H' is of the form $\sigma(\kappa, \lambda)$ for $\kappa \in S$ and λ fulfilling (A1) and (A2). By separability of H , \mathcal{F} is countable. By Step 1, \mathcal{F} spans H_0 . It will suffice, therefore, to represent each $\sigma(\kappa, \lambda)$ with λ fulfilling (A1) and (A2) as a finite direct sum of representations $\sigma(\kappa, \lambda')$, where λ' fulfils (A1), (A2), and (A3).

Fix $\kappa = (k_1, \dots, k_n)$ and λ with (A1) and (A2). Let S_κ consist of all permutations ρ of $\{1, \dots, n\}$ such that for $1 \leq i, j \leq n$

$$(10) \quad \rho(i) = j \implies k_i = k_j.$$

For $\rho \in S_\kappa$, let

$$X^\rho = \{(x_1, \dots, x_n) \in X^n : x_{\rho(i)} <_X x_{\rho(j)} \text{ for all } i, j \text{ with } k_i = k_j\}.$$

Note that $L^2(\lambda \upharpoonright X^\rho)$ is invariant under the representation $\sigma(\kappa, \lambda)$. Since λ fulfils (A2), it follows that $\sigma(\kappa, \lambda)$ is the direct sum of the representations $\sigma(\kappa, \lambda \upharpoonright X^\rho)$ with ρ varying over S_κ . For $\rho \in S_\kappa$, let $\bar{\rho}: X^\rho \rightarrow X^{\text{id}}$, where id is the identity permutation, be given by

$$\bar{\rho}(x_1, \dots, x_n) = (x_{\rho(1)}, \dots, x_{\rho(n)}).$$

From (10) it is clear that $\sigma(\kappa, \lambda \upharpoonright X^\rho)$ can be replaced by $\sigma(\kappa, \bar{\rho}_*(\lambda) \upharpoonright X^{\text{id}})$, and so $\sigma(\kappa, \lambda)$ is the direct sum of the representations $\sigma(\kappa, \bar{\rho}_*(\lambda) \upharpoonright X^{\text{id}})$ with ρ varying over S_κ . It is also clear that the measure $\bar{\rho}_*(\lambda) \upharpoonright X^{\text{id}}$ fulfils (A3), as well as (A2) and (A1). Thus, the conclusion follows.

Step 3. We now assume that the representation ϕ when restricted to H_0 is the direct sum as described in Step 2. We show how to modify this direct sum so that it fulfils (A4) as in the conclusion of the theorem. Fix $\kappa \in S$. Let λ^j , $j < m$, list all non-zero measures on $X^{|\kappa|}$ appearing in $\sigma(\kappa, \lambda^j)$ in the direct sum given by Step 2. Here $m \in \mathbb{N} \cup \{\infty\}$. We can, and we do, assume that each λ^j is a probability measure. We will use the following general and easy observation. Assume we have finite Borel measures

$$\nu_{i-1} \ll \dots \ll \nu_2 \ll \nu_1$$

on a standard Borel space and another finite Borel measure μ on the same space. Then $\mu = \mu_1 + \cdots + \mu_i$, where

$$\begin{aligned} \mu_j &\ll \nu_{j-1}, \text{ for } 2 \leq j \leq i; \\ \mu_j &\perp \nu_j, \text{ for } 1 \leq j \leq i-1; \\ \mu_j &\perp \mu_{j'}, \text{ for } 1 \leq j, j' \leq i, j \neq j'. \end{aligned}$$

Using this observation, by induction on i , we find finite Borel measures λ_j^i with $j \leq i$ so that the following conditions hold

- (a) $\lambda^i = \lambda_1^i + \cdots + \lambda_i^i$;
- (b) $\lambda_i^i \ll \cdots \ll \lambda_2^i + \cdots + \lambda_2^i \ll \lambda_1^i + \lambda_1^i + \cdots + \lambda_1^i$;
- (c) $\lambda_j^i \perp (\lambda_j^j + \cdots + \lambda_j^{i-1})$, for $j < i$;
- (d) $\lambda_j^i \perp \lambda_{j'}^i$, for $j, j' \leq i, j \neq j'$.

Note that condition (b) for $i-1$ is used as an inductive assumption and is maintained in the induction by the first condition in the general observation above.

Let now λ_κ^j for $j < m$ be the measure

$$\lambda_j^j + 2^{-1}\lambda_j^{j+1} + 2^{-2}\lambda_j^{j+2} + \cdots.$$

This is a finite measure by (a) since each λ^i was assumed to be a probability measure. Set also $\lambda_\kappa^j = 0$ for $j \in \mathbb{N}$ and $j \geq m$. By conditions (a) and (d), the direct sum of the representations $\sigma(\kappa, \lambda^i)$ for $i \in \mathbb{N}$ is the direct sum of the representations $\sigma(\kappa, \lambda_j^i)$ for $j \leq i, i \in \mathbb{N}$. By condition (c) and the definition of λ_κ^j , this latter direct sum is also the direct sum of the representations $\sigma(\kappa, \lambda_\kappa^j)$ for $j \in \mathbb{N}$. Condition (b) ensures that $\lambda_\kappa^{j'} \ll \lambda_\kappa^j$ for $j' > j$, that is, (A4) holds. Thus, the measures λ_κ^j , $\kappa \in S$, $j \in \mathbb{N}$, are as required.

Proof of uniqueness of the presentation. For subsets P, Q of X , we write

$$P <_X Q$$

if $x <_X y$ for all $x \in P$ and $y \in Q$.

We will use the following elementary observation, whose justification we leave to the reader.

Observation. Let $\kappa = (k_1, \dots, k_m)$ and $\kappa' = (l_1, \dots, l_n)$ be in S . Let $q \in \mathbb{N}$ and let

$$u: \{1, \dots, m\} \rightarrow \{1, \dots, q\} \text{ and } v: \{1, \dots, n\} \rightarrow \{1, \dots, q\}$$

be injective. Assume that for all $z_1, \dots, z_q \in \mathbb{T}$, we have

$$z_{u(1)}^{k_1} \cdots z_{u(m)}^{k_m} = z_{v(1)}^{l_1} \cdots z_{v(n)}^{l_n}.$$

Then $m = n$ and for each $r \in \{1, \dots, q\}$

$$\{u(i) : k_i = r\} = \{v(i) : l_i = r\};$$

therefore, since $\kappa, \kappa' \in S$, $\kappa = \kappa'$. It follows that, if additionally for all $i < j \leq m$

$$k_i = k_j \Rightarrow u(i) < u(j) \quad \text{and} \quad l_i = l_j \Rightarrow v(i) < v(j),$$

then $u = v$.

We will also consider X equipped with the order topology induced by $<_X$, which is assumed to be second countable and compact.

Assume we are given a representation ϕ of $L^0(\mu, \mathbb{T})$ on a separable Hilbert space H . Assume that we have two presentations of the restriction of ϕ to H_0 given by λ_κ^j and by $(\lambda')_\kappa^j$, for $i \in \mathbb{N}$ and $\kappa \in S$. Assume further towards a contradiction that these two presentations do not fulfill the uniqueness criterion from the theorem. Thus, there exist κ , j , and a Borel set $K \subseteq X^{|\kappa|}$ whose measure is positive with respect to one of the measures λ_κ^j , $(\lambda')_\kappa^j$ and is zero with respect to the other. Fix such a κ and such a j . They will be called κ_0 and j_0 , respectively. Let κ_0 be equal to (k_1, k_2, \dots, k_n) ; in particular, $|\kappa_0| = n$. Without loss of generality, we can assume that for all $j < j_0$, we have $\lambda_{\kappa_0}^j \sim (\lambda')_{\kappa_0}^j$, that

$$(11) \quad \lambda_{\kappa_0}^{j_0}(K) > 0 \quad \text{and} \quad (\lambda')_{\kappa_0}^{j_0}(K) = 0,$$

and, by using (A2) and (A3), that

$$(12) \quad K \subseteq \{x \in X^n : x_1 <_X x_2 <_X \dots <_X x_n\}.$$

We can also assume, by shrinking K if necessary, that K is compact in the product topology on X^n .

A partition \mathcal{P} of X into Borel sets will be called *admissible* if $P <_X Q$ or $Q <_X P$ for distinct $P, Q \in \mathcal{P}$ and

$$K \subseteq \bigcup_u \prod_{i=1}^n u(i),$$

where u varies over the set of all functions $u : \{1, \dots, n\} \rightarrow \mathcal{P}$ with

$$(13) \quad u(i_1) <_X u(i_2), \quad \text{for } 1 \leq i_1 < i_2 \leq n.$$

Using (12) and compactness of K , we see that there exists an admissible partition. Note that a partition finer than an admissible one is also admissible. Further, for an admissible partition \mathcal{P} and $j \in \mathbb{N}$, put

$$U_{\mathcal{P}}^j = \{u : \{1, \dots, n\} \rightarrow \mathcal{P} : u \text{ fulfills (13) and } \lambda_{\kappa_0}^j(K \cap \prod_{i=1}^n u(i)) > 0\}.$$

With a sequence $z_{\mathcal{P}} = (z_P \in \mathbb{T} : P \in \mathcal{P})$, we associate the unitary operator

$$(14) \quad A_{\mathcal{P}, z_{\mathcal{P}}} = \phi\left(\sum_{P \in \mathcal{P}} z_P \chi_P\right).$$

Define H_K to be the closure of the set of all $h \in H$ with the following property. For every admissible partition \mathcal{P} of X , one can represent h as

$$(15) \quad h = \sum_{u \in U_{\mathcal{P}}} h_u,$$

where, for every $z_{\mathcal{P}}$, h_u is an eigenvector of $A_{\mathcal{P}, z_{\mathcal{P}}}$ with eigenvalue

$$(16) \quad \prod_{i=1}^n z_{u(i)}^{k_i}.$$

We claim that when ϕ is viewed as the direct sum of $\sigma(\kappa, \lambda_{\kappa}^j)$ for $j \in \mathbb{N}$ and $\kappa \in S$, then H_K is the direct sum of $L^2(\lambda_{\kappa_0}^j \upharpoonright K)$ over $j \in \mathbb{N}$, and when ϕ is viewed as the direct sum of $\sigma(\kappa, (\lambda')_{\kappa}^j)$, then H_K is the direct sum of $L^2((\lambda')_{\kappa_0}^j \upharpoonright K)$ over $j \in \mathbb{N}$. We write out a proof only for λ_{κ}^j . In it, we identify H with the direct sum of $L^2(\lambda_{\kappa}^j)$ over $j \in \mathbb{N}$ and $\kappa \in S$. That the direct sum of $L^2(\lambda_{\kappa_0}^j \upharpoonright K)$, $j \in \mathbb{N}$, $\kappa \in S$, is included in H_K is not difficult to check and we leave it to the reader. We show the other inclusion. Assume for contradiction that it does not hold. Then we have an element h of H_K whose projection on $L^2(\lambda_{\kappa}^j)$ is non-zero for some j and some $\kappa \neq \kappa_0$ or whose projection on $L^2(\lambda_{\kappa_0}^j)$, for some j , has support not included in K . Let $A \subseteq X^{|\kappa|}$ be the support of this projection. Set $m = |\kappa|$ and $\kappa = (l_1, \dots, l_m)$. By condition (A2) and by compactness of K , there is an admissible partition \mathcal{P} of X and an injective function $v: \{1, \dots, m\} \rightarrow \mathcal{P}$ such that

$$(17) \quad \lambda_{\kappa}^j(A \cap (v(1) \times \dots \times v(m))) > 0,$$

and either $\kappa \neq \kappa_0$ or ($\kappa = \kappa_0$ and $v \notin U_{\mathcal{P}}^j$). Note that, by (A3) and by (17), v satisfies for all $1 \leq i_1 < i_2 \leq m$

$$(18) \quad l_{i_1} = l_{i_2} \implies v(i_1) <_X v(i_2).$$

Now if h is represented as a sum as in (15) for the \mathcal{P} found above, then there exists an h_u , for some $u \in U_{\mathcal{P}}$, whose projection on $L^2(\lambda_{\kappa}^j)$ has support intersecting $v(1) \times \dots \times v(m)$ on a set of λ_{κ}^j positive measure. Now, h_u is an eigenvector of (14) for every $z_{\mathcal{P}}$. Its eigenvalue for a given $z_{\mathcal{P}}$ must be equal to

$$\prod_{i=1}^m z_{v(i)}^{l_i}$$

since every value of a function from $L^2(\lambda_{\kappa}^j)$ attained on $v(1) \times \dots \times v(m)$ is multiplied by that number when the function is acted on by (14). On

the other hand, this eigenvalue is also equal to (16) for the u found above. Now, using the observation from the beginning of the proof of uniqueness and using (18) and (13) for u , we see that $\kappa = \kappa_0$ and $v = u$, so $v \in U_{\mathcal{P}}^j$, contradiction. Thus, H_K is the direct sum of $L^2(\lambda_{\kappa_0}^j \upharpoonright K)$ over $j \in \mathbb{N}$. Similarly, we get that it is the direct sum of $L^2((\lambda')_{\kappa_0}^j \upharpoonright K)$ over $j \in \mathbb{N}$.

Using the presentation of H_K as a direct sum with respect to $(\lambda')_{\kappa}^j$, $\kappa \in S$, $j \in \mathbb{N}$, we show that there are $j_0 - 1$ vectors such that H_K is the smallest closed subspace containing these vectors and invariant under $L^0(\mu, \mathbb{T})$. Take a copy of χ_K in each $L^2((\lambda')_{\kappa_0}^j)$ for $j < j_0$. Note that vectors obtained from each of the $j_0 - 1$ copies of χ_K by acting on them by elements of $L^0(\mu, \mathbb{T})$ separate points of K . So the smallest closed subspace H' containing all these vectors contains each $L^2((\lambda')_{\kappa_0}^j \upharpoonright K)$ for $j < j_0$. Since $(\lambda')_{\kappa_0}^j(K) = 0$ for $j \geq j_0$, we see that H' contains the direct sum of $L^2((\lambda')_{\kappa_0}^j \upharpoonright K)$ over all j , that is, by what was proved above, it is equal to H_K .

On the other hand, using the presentation of H_K as a direct sum with respect to λ_{κ}^j , $\kappa \in S$, $j \in \mathbb{N}$, we show that given $j_0 - 1$ vectors f_j , $j < j_0$, in H_K the closed subspace H' spanned by all the vectors obtained from f_j , $j < j_0$, by acting on them by $L^0(\mu, \mathbb{T})$ is a proper subspace of H_K . First note that since H_K is the direct sum of $L^2(\lambda_{\kappa_0}^j \upharpoonright K)$, for $j \in \mathbb{N}$, with each element $h \in H_K$ we can associate a sequence h^j , $j \in \mathbb{N}$, with each h^j being a $\lambda_{\kappa_0}^j$ class of a function from K to \mathbb{C} . Since $\lambda_{\kappa_0}^{j_0} \ll \lambda_{\kappa_0}^j$ for $j \leq j_0$, we see that h^j , for $j \leq j_0$, determines a single $\lambda_{\kappa_0}^{j_0}$ function class, which we again denote by h^j . Then we define $\bar{h} \in L^2(\lambda_{\kappa_0}^{j_0} \upharpoonright K, \mathbb{C}^{j_0})$ by letting for $\lambda_{\kappa_0}^{j_0}$ almost every $x \in K$

$$\bar{h}(x) = (h^1(x), \dots, h^{j_0}(x)) \in \mathbb{C}^{j_0}.$$

Now, for $\lambda_{\kappa_0}^{j_0}$ almost every $x \in K$, let V_x be the linear subspace of \mathbb{C}^{j_0} spanned by $\overline{f_1}, \dots, \overline{f_{j_0-1}}$. Note that the function $K \ni x \rightarrow V_x$ is $\lambda_{\kappa_0}^{j_0}$ measurable and that the dimension of V_x does not exceed $j_0 - 1$. Observe that for $g \in L^0(\mu, \mathbb{T})$ and $h \in H_K$,

$$\overline{\phi(g)(h)}(x) = z_x \bar{h}$$

for some $z_x \in \mathbb{C}$ for $\lambda_{\kappa_0}^{j_0}$ almost every $x \in K$. It follows that for each $h \in H'$, we have

$$(19) \quad \bar{h}(x) \in V_x, \text{ for } \lambda_{\kappa_0}^{j_0} \text{ almost every } x \in K.$$

Since the dimension of V_x is less than that of \mathbb{C}^{j_0} , we have that $\mathbb{C}^{j_0} \setminus V_x$ is non-empty for $\lambda_{\kappa_0}^{j_0}$ almost every $x \in K$. Thus, by the Jankov–von Neumann selection theorem, see [6, Theorem 18.1], there is a bounded $\lambda_{\kappa_0}^{j_0}$ measurable function $F: K \rightarrow \mathbb{C}^{j_0}$ such that

$$(20) \quad F(x) \notin V_x, \text{ for } \lambda_{\kappa_0}^{j_0} \text{ almost every } x \in K.$$

We easily see that there is $f \in H_K$ with

$$\overline{f} = F.$$

It follows from (19), (20), and the fact that $\lambda_{\kappa_0}^{j_0}(K) > 0$ that f is not in H' . Thus, H' is a proper subspace of H_K . This conclusion yields a contradiction and completes the proof of uniqueness of the presentation.

3. UNITARY REPRESENTATIONS OF $C(M, \mathbb{T})$

In this section, M is a second countable, compact, zero-dimensional space. We also fix a linear order $<_M$ on M such that the order topology induced by it is compact and second countable and has the same Borel sets as the original topology on M . In fact, since M is zero-dimensional, it can be viewed as a subset of $\{0, 1\}^{\mathbb{N}}$, see [6, Theorem 7.8], and the order $<_M$ can be defined to be the pull-back of the lexicographic order on $\{0, 1\}^{\mathbb{N}}$. Then the order topology induced by $<_M$ is equal to the original topology on M .

3.1. Description of representations. The description here is essentially the one from Subsection 2.1 except that, obviously, we do not have a condition analogous to (A1). We keep the piece of notation S standing for the set of all $\kappa = (k_1, \dots, k_n)$ of elements of $\mathbb{Z} \setminus \{0\}$ with $k_1 \leq k_2 \leq \dots \leq k_n$.

Given $\kappa = (k_1, \dots, k_n) \in S$ and a finite Borel measure λ on M^n , we consider the representation of $C(M, \mathbb{T})$ on $L^2(\lambda, \mathbb{C})$ given by:

$$C(M, \mathbb{T}) \ni f \rightarrow U_f \in \mathcal{U}(L^2(\lambda, \mathbb{C})),$$

where for $h \in L^2(\lambda, \mathbb{C})$

$$U_f(h) = \left(\prod_{i \leq n} (f \circ \pi_i)^{k_i} \right) h,$$

where, for $i \leq n$, $\pi_i: M^n \rightarrow M$ is the projection on the i -th coordinate. We denote this representation again by $\sigma(\kappa, \lambda)$.

For $\kappa \in S$ with $n = |\kappa|$ and a finite Borel measure λ on M^n , we will consider the following two conditions: for all $1 \leq i < j \leq n$

$$(B1) \quad \lambda(\{(x_1, \dots, x_n) \in M^n : x_i = x_j\}) = 0$$

and, for $1 \leq i < j \leq n$ with $k_i = k_j$,

$$(B2) \quad \lambda(\{(x_1, \dots, x_n) \in M^n : x_j <_M x_i\}) = 0.$$

3.2. Statement of the theorem.

Theorem 3.1. *Let M be a compact, second countable, zero-dimensional space. Let ϕ be a continuous unitary representation of $C(M, \mathbb{T})$ on a separable complex Hilbert space H . Consider H_0 , the orthogonal complement of*

$$\{v \in H : \forall f \in C(M, \mathbb{T}) \phi(f)(v) = v\}.$$

Existence. For each $\kappa \in S$, there exist finite Borel measures λ_κ^i , $i \in \mathbb{N}$, on $M^{|\kappa|}$ with properties (B1), (B2), and with

$$(B3) \quad \lambda_\kappa^j \ll \lambda_\kappa^i, \text{ for } i < j,$$

such that the representation ϕ restricted to H_0 is the direct sum of the representations $\sigma(\kappa, \lambda_\kappa^i)$ with $\kappa \in S$ and $i \in \mathbb{N}$.

Uniqueness. If the restriction of ϕ to H_0 is presented as the direct sum of $\sigma(\kappa, (\lambda')_\kappa^i)$, for $\kappa \in S$, $i \in \mathbb{N}$, with (B1), (B2), and (B3), then λ_κ^i and $(\lambda')_\kappa^i$ are absolutely continuous with respect to each other for all $\kappa \in S$ and $i \in \mathbb{N}$.

It is not difficult to see that the theorem above fails without the assumption of zero-dimensionality, for example, it fails for $M = [0, 1]$.

We have a corollary similar to Corollary 2.2. Recall that given a finite Borel measure ν on a compact second countable space M , mapping a function from $C(M, \mathbb{T})$ to its equivalence class in $L^0(\nu, \mathbb{T})$ induces a continuous homomorphism from the first group to the latter.

Corollary 3.2. *Let M be a compact second countable zero-dimensional space. Let ϕ be a continuous unitary representation of $C(M, \mathbb{T})$. There exists a finite Borel measure ν on M such that*

- (i) *ϕ is the composition of the natural homomorphism from $C(M, \mathbb{T})$ to $L^0(\nu, \mathbb{T})$ and a continuous unitary representation of $L^0(\nu, \mathbb{T})$;*
- (ii) *if ν' is a finite Borel measure on M such that ϕ is the composition of the natural homomorphism from $C(M, \mathbb{T})$ to $L^0(\nu', \mathbb{T})$ and a continuous unitary representation of $L^0(\nu', \mathbb{T})$, then $\nu \ll \nu'$.*

Before proving the corollary we recall a simple lemma that will also be used in the proof of Theorem 3.1 and whose proof we leave to the reader.

Lemma 3.3. *Let ν be a finite Borel measure on a compact second countable space M . The image of the natural homomorphism from $C(M, \mathbb{T})$ to $L^0(\nu, \mathbb{T})$ is dense in $L^0(\nu, \mathbb{T})$.*

Proof of Corollary 3.2. This corollary follows from Theorem 3.1 in a manner essentially identical to the argument showing Corollary 2.2. The only addition is the following remark. After the measure ν is produced, we get that the representation ϕ remains continuous when $C(M, \mathbb{T})$ is taken with

the L^0 topology with respect to ν . At this point we use Lemma 3.3 to extend ϕ to a continuous unitary representation of $L^0(\nu, \mathbb{T})$. After that the proof again follows the route of the proof of Corollary 2.2. \square

We will give the proof of Theorem 3.1 in Subsection 3.4. In Subsection 3.3, we collect some more results needed for the argument.

3.3. More background results. As explained below the following result is a combination of the factorization theorems from [12], [10] and [16].

Proposition 3.4. *Let M be a compact, second countable space, and let μ be a Borel probability measure on a standard Borel space. Each continuous operator from $C(M, \mathbb{R})$ to $L^0(\mu, \mathbb{R})$ factors through a real Hilbert space.*

By [12] each operator $C(M, \mathbb{R}) \rightarrow L^0(\mu, \mathbb{R})$ factors through $L^p(\mu, \mathbb{R})$ for each $0 \leq p < 1$; by [10, Théorème III,2], for each $0 < p < 1$, each operator $C(M, \mathbb{R}) \rightarrow L^p(\mu, \mathbb{R})$ factors through $L^q(\mu, \mathbb{R})$ for some $1 < q \leq 2$; by [16, Corollary 4.4], for each $1 \leq q \leq 2$, each operator $C(M, \mathbb{R}) \rightarrow L^q(\mu, \mathbb{R})$ factors through a Hilbert space. A simple concatenation of these three results gives the proposition above. It is possible that the proposition can be obtained by the methods of [8, Corollaire 34, Théorème 93], where the same result is proved with $C(M, \mathbb{R})$ replaced by L^∞ .

We will need the following result that is a combination of theorems of Grothendieck and Pietsch, see [16, Theorem 5.4(b), Corollary 1.5].

Proposition 3.5. *Let M be a compact, second countable space and let H be a separable real Hilbert space. Let $T: C(M, \mathbb{R}) \rightarrow H$ be a continuous linear operator. There exists a Borel probability measure ν on M such that T remains continuous if we take $C(M, \mathbb{R})$ with the L^2 topology with respect to ν .*

3.4. Proof of Theorem 3.1. As before, when we say “representation” we mean “strongly continuous unitary representation.”

Assume we have a representation ϕ of $C(M, \mathbb{T})$ on a separable complex Hilbert space. As in the start of the proof of Theorem 2.1, we use the spectral theorem, Proposition 2.4, to see that the unitary representation ϕ of $C(M, \mathbb{T})$ is the form

$$C(M, \mathbb{T}) \ni f \rightarrow N_f \in \mathcal{U}(L^2(\nu, \mathbb{C}))$$

where

$$N_f(h) = \psi(f) \cdot h$$

for some continuous homomorphism

$$(21) \quad \psi: C(M, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T}).$$

Consider now the exponential map

$$C(M, \mathbb{R}) \rightarrow C(M, \mathbb{T}); f \rightarrow \exp(2\pi i f).$$

Note that $C(M, \mathbb{T})$ is connected since M is zero-dimensional. Since the image of $C(M, \mathbb{R})$ under the exponential map contains an open ball around the identity in $C(M, \mathbb{T})$, by connectedness of $C(M, \mathbb{T})$, it is actually equal to $C(M, \mathbb{T})$. Thus, we have a surjective continuous homomorphism

$$(22) \quad C(M, \mathbb{R}) \rightarrow C(M, \mathbb{T}); f \rightarrow \exp(2\pi i f).$$

By precomposing ψ with the exponential homomorphism (22), we obtain a continuous homomorphism $\psi': C(M, \mathbb{R}) \rightarrow L^0(\nu, \mathbb{T})$. By Proposition 2.3, there exists a continuous linear operator

$$\theta: C(M, \mathbb{R}) \rightarrow L^0(\nu, \mathbb{R})$$

such that

$$\psi' = \exp(2\pi i \theta).$$

By Proposition 3.4, θ factors through a Hilbert space, that is, there is a continuous linear operator $\theta': C(M, \mathbb{R}) \rightarrow H$, where H is a separable real Hilbert space, such that θ is the composition of θ' and a continuous linear operator $H \rightarrow L^0(\nu, \mathbb{R})$. We can assume that H is separable since $C(M, \mathbb{R})$ is. Now by Proposition 3.5, there exists a probability Borel measure μ on M such that θ' remains continuous if $C(M, \mathbb{R})$ is taken with the L^2 topology with respect to μ .

We make two observations. First, since M is zero-dimensional, for each $f \in C(M, \mathbb{T})$, there exists $g \in C(M, \mathbb{R})$ with $|g| < 2/3$ and $f = \exp(2\pi i g)$. Our second observation consist of noticing that the L^2 and the L^0 topologies with respect to μ are identical on the subset of $C(M, \mathbb{R})$ consisting of all functions bounded by $2/3$. Now, given a sequence (f_n) in $C(M, \mathbb{T})$ convergent to 1 in the L^0 topology with respect to μ , pick a sequence (g_n) in $C(M, \mathbb{R})$ such that $|g_n| < 2/3$ and $f_n = \exp(2\pi i g_n)$ for each n . It follows that (g_n) converges to 0 in the L^0 topology with respect to μ . Thus, we get that (g_n) converges to 0 in the L^2 topology, which implies that the sequence $(\psi'(g_n))$ converges to 1 in $L^0(\nu, \mathbb{T})$. Since, by definition of ψ' , $\psi'(g_n) = \psi(f_n)$, we see that $(\psi(f_n))$ converges to 1 in $L^0(\nu, \mathbb{T})$. It follows that the homomorphism ψ given by (21) remains continuous if $C(M, \mathbb{T})$ is taken with the L^0 topology with respect to μ .

By Lemma 3.3 we can extend ψ to a continuous homomorphism

$$\tilde{\psi}: L^0(\mu, \mathbb{T}) \rightarrow L^0(\nu, \mathbb{T}).$$

We apply now Theorem 2.1 to the representation induced by this homomorphism and the existence part of Theorem 3.1 follows immediately.

The uniqueness part is a consequence of the uniqueness part of Theorem 2.1. Assume we have two presentations as in Theorem 3.1 given by (λ_κ^i) and $((\lambda')_\kappa^i)$ for $\kappa \in S$ and $i \in \mathbb{N}$ of a single representation ϕ of $C(M, \mathbb{T})$ restricted to H_0 . We define a finite Borel measure μ on M as follows. For each λ_κ^i consider all finitely many measures on M obtained from λ_κ^i by pushing it forward by all the projections from $M^{|\kappa|}$ to M . Collect all such push-forwards of λ_κ^i for all i and κ and form a weighted sum of this countable collection of finite Borel measures on M obtaining μ . In the same manner define μ' from $((\lambda')_\kappa^i)$ for $i \in \mathbb{N}$ and $\kappa \in S$. Now it is clear, using either one of the two presentations, that the representation ϕ remains continuous when we consider $C(M, \mathbb{T})$ with the L^0 topology with respect to the measure $\mu + \mu'$. Using density of $\widetilde{C}(M, \mathbb{T})$ in $L^0(\mu + \mu', \mathbb{T})$, we see that ϕ extends to a unitary representation $\widetilde{\phi}$ of $L^0(\mu + \mu', \mathbb{T})$. Now both presentation, the one given by (λ_κ^i) and the one given by $((\lambda')_\kappa^i)$, are presentations of $\widetilde{\phi}$ restricted to H_0 as in Theorem 2.1. By the uniqueness part of that theorem, we get $\lambda_\kappa^i \sim (\lambda')_\kappa^i$ for all i and κ as required.

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